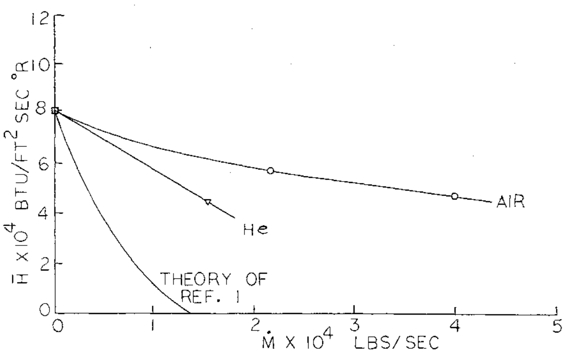

Fig. 4 Reduction in heat-transfer coefficient with injection.

attached boundary layer calculated by the method of Ref. 7. This result agrees with the predictions of Chapman.

The heat-transfer distributions with gas injection are shown in Fig. 2. Data were obtained for four cases: 1) air injection from the upstream cavity wall ($\dot{M} = 2.26 \times 10^{-4}$ lb/sec), 2) air injection from the downstream cavity wall, ($\dot{M} = 2.15 \times 10^{-4}$ lb/sec), 3) air injection from both walls simultaneously ($\dot{M} = 4.00 \times 10^{-4}$ lb/sec), and 4) helium injection from both walls simultaneously ($\dot{M} = 1.56 \times 10^{-4}$ lb/sec). These results indicate that gas injection into a region of laminar separation reduces the large heat-transfer rate that occurs in the downstream reattachment region and yields a more uniform heat-transfer distribution. In addition, the manner in which the coolant gas is introduced into the cavity (i.e., upstream, downstream, etc.) has little influence on the heat-transfer distribution.

In Fig. 3, the results are shown for the convective heat-transfer coefficient based upon a recovery factor equal to the square root of the Prandtl number. This has been suggested as a reasonable recovery factor for laminar-separated flow without gas injection.[1, 5] It is clear that helium is a much more effective coolant than air. The greatest reductions in heat transfer occurred in the reattachment region of the cavity with little if any reductions in the upstream portions of the cavity.

In Fig. 4, the reduction of the average heat-transfer coefficient with injection of air or helium is shown. These results are compared to the predictions of Chapman[1] for the case $M = \infty$. The reductions in heat transfer attained in the present study with air and helium as coolants, though substantial, are not as great as predicted by theory. Whereas the theory of Ref. 1 considers only air injection, the present study indicates that helium is a more effective coolant than air.

The following comments and conclusions can be made from the preceding results:

1) The heat transfer to a region of laminar separated flow can be substantially reduced by gas injection.

2) Chapman's theory appears to overestimate the effectiveness of mass injection.

3) The greatest reductions in heat transfer occur in the regions where heat transfer is greatest. This results in more uniform distributions of heat transfer in the cavity.

4) The injection of a lightweight gas of high specific heat, such as helium, leads to greater reductions in the heat transfer than does the same quantity of air injection.

5) The position of the gas injection had little influence upon the heat-transfer distribution that was obtained.

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An Estimate of the Decay of Turbulent Intensity in a Shear Flow

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Introduction

A CONSIDERABLE effort is being expended currently to obtain estimates of the radar cross section of ionized turbulent wakes. The simplest case prevails when the plasma is very tenuous, the radar illuminating frequency is much larger than the plasma frequency of the ionized gas, and the effects of multiple scattering can be ignored. When all of these conditions are met, the radar cross section can be calculated by the first Born approximation. The calculated cross section is proportional to the mean value of the square of the electron-density fluctuations and the three-dimensional Fourier transform of the correlation function (spectrum) of the effective dielectric constant of the plasma.

A previous paper[1] predicted the radar cross section of a turbulent underdense wake based on the following three assumptions: 1) the spectrum of the fluctuations of the index of refraction of the plasma was assumed to be proportional to the turbulent energy spectrum, 2) a Kolmogoroff spectrum was assumed to prevail, and 3) the turbulent energy was assumed constant. In this paper the assumption of constant energy is examined in some detail.

Analysis

The turbulent energy for constant density is given by the temporal mean, which we assume to be equal to the ensemble average

$$\langle E \rangle = \tfrac{1}{2} \langle u^2 + v^2 + w^2 \rangle$$

where, for convenience, we have set the density $\rho \equiv 1$, and u, v, and w are the components of the turbulent velocity fluctuations. Since we shall deal only with mean values, we shall drop the angular brackets, it being understood that temporal mean values are to be taken.

The turbulent energy E is assumed here to be a known function of the wave number κ. In the development that follows, we assume that, throughout the process of decay, the energy spectrum $E(\kappa)$ remains self-similar, and only the amplitude decreases uniformly over all of the frequencies.

The time rate of change of the energy in a flowing medium is given by the substantial derivative

$$DE/Dt \equiv (\partial E/\partial t) + \bar{q} \cdot \nabla E$$

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where $\bar{q} = \hat{\imath}U + \hat{\jmath}V + \hat{k}W$, and U, V, and W are the velocity components of the mean flow.

The changes in turbulent energy are attributable to three causes. The first is the decay of individual turbulence elements. (For the sake of brevity, we shall refer to such elements as eddies, but this does not imply that they are to be considered akin to isolated vortices.) The second is the generation of turbulence owing to the work done by the turbulent shear stresses. The third is the diffusion of turbulence owing to the presence of turbulent intensity gradients (we shall neglect this effect).

Energy-decay model

We assume that the energy is distributed among the eddies in a consistent manner and assert that this distribution can be characterized by $l = \kappa_E^{-1}$, where κ_E is the wave number at which $E(\kappa)$ is a maximum. Alternatively, l may be associated with the turbulent macroscale L, which is given by

$$L^{-1} = \int_0^\infty E(\kappa)\kappa\, d\kappa \Big/ \int_0^\infty E(\kappa)\, d\kappa$$

or with the correlation length defined as, for example, the e-folding distance of the correlation function. For the purpose of estimating the decay of the turbulent energy, we assume the eddies to behave like solid spheres of radius l. This is equivalent to replacing the spectral representation of $E(\kappa)$ by a single spectral element $E(\kappa_E)$.

Decay of turbulence

The resistance that an eddy experiences is basically pressure drag, and, therefore, is proportional to the square of its relative velocity u with respect to the mean flow and to its cross-sectional area. Thus the resistance D varies as l^2u^2, where l is some characteristic turbulence scale. The work done per unit time per unit volume is given by

$$Du/l^3 \sim [(l^2u^2)u/l^3] = u^3/l$$

If we assume now that the turbulence is approximately isotropic, then $u^2 = v^2 = w^2 \approx 2E/3$. Thus

$$DE/Dt|_1 = -CE^{3/2}/l$$

where C is a constant of proportionality.

Generation of turbulent energy

By assuming that all of the dissipated flow energy is transformed into turbulent energy, we obtain from the energy equation, in the absence of heat transfer,

$$DE/Dt|_2 = \nu_T(dU/dy)^2$$

where ν_T is the turbulent diffusivity. We have also assumed that all of the terms in the dissipation function except dU/dy are negligible, i.e., the flow is of the boundary-layer type. The sum of the generation term and the dissipation term gives the total change in turbulent energy, i.e.,[2]

$$DE/Dt = (-CE^{3/2}/l) + \nu_T(dU/dy)^2$$

We assume that the steady state prevails; thus, the left-hand side reduces to $U\ dE/dx$. We set the turbulent diffusivity ν_T equal to $Kl(E)^{1/2}$, and we assume the velocity gradient to vary inversely with the downstream distance. To avoid a singularity at $x = 0$, we set $dU/dy = A/(x + x_0)$. Thus

$$U(dE/dx) = (-CE^{3/2}/l) + [KlA^2/(E)^{1/2}/(x + x_0)^2]$$

We introduce the nondimensional variables $\epsilon = (E)^{1/2}/U_0$ and $\xi = x + x_0/l$ and divide through by ϵ. Thus

$$(2U/U_0)(d\epsilon/d\xi) = -C\epsilon^2 + [K(A/U_0)^2/\xi^2]$$

(1) *Constant velocity:* For the case U/U_0 = constant, for example, unity and $A = aU_0$, a closed-form solution can be obtained. By setting

$$\eta = (C/2)\xi\epsilon \qquad \text{and} \qquad a^2KC/4 = q$$

we obtain

$$\xi(d\eta/d\xi) + \eta^2 - \eta - q = 0$$

This equation is separable, and the solution can be obtained in terms of an elementary integral. For the case $q \ll 1$ and making use of the fact that $C \approx K^3$ (Ref. 2), we obtain the simple result

$$(K^3/2)\epsilon = 1/\xi$$

In other words, the turbulent energy drops off as the inverse of the normalized distance squared.

It is interesting to note that this energy-decay law is the same as the one found by Bateman[3] for the decay of a simple laminar vortex. Bateman's analysis, incidentally, is an exact solution of the incompressible Navier-Stokes equations.

2) *Wake-like flows:* We assume that Townsend's[4] asymptotic wake velocity-decay law is applicable; i.e.,

$$(U/U_0)^3 = (1/144)R_T'^2[C_DA/(x + x_0)^2]$$

where C_DA refers to the object generating the wake, and where we have taken as the denominator $(x + x_0)^2$ rather than $(x - x_0)^2$. The latter change is really only a convenience to avoid a singularity at $x = x_0$.

Using Townsend's numerical values and setting

$$l = \alpha^{-3/2}(C_DA)^{1/2} \qquad R_T' = 14.1 \qquad \text{and} \qquad C = K^3$$

we obtain

$$2.226\ \alpha(d\epsilon/d\xi) = -K^3\xi^{2/3}\epsilon^2 + Ka^2\xi^{-4/3}$$

This equation was solved numerically† for $K = 0.4$, $a = 1$, and $2.226\ \alpha = 0.01$, 0.1, 1.0, and 10. The results are presented in Fig. 1. For

$$l = 0.1\ (C_DA)^{1/2} \qquad C = 10.45$$
$$l = 0.5\ (C_DA)^{1/2} \qquad C = 3.55$$
$$l = (C_DA)^{1/2} \qquad C = 2.226$$

Discussion and Results

The first thing to note is that, for the case of constant velocity, the turbulent energy is a monotonically decreasing function of the axial distance. For wake-like flows, the turbulent energy increases at first and then decays. The behavior of the energy decay is very sensitive to the assumed

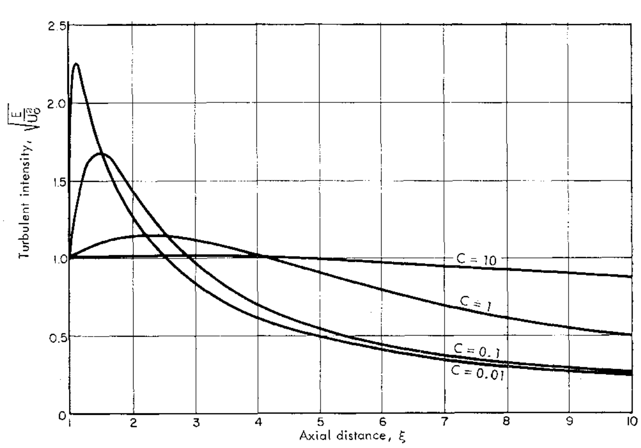

Fig. 1 Turbulent intensity vs distance for C = 10, 1.0, 0.1, and 0.01.

† The author wishes to acknowledge the help of Marion Fickett of the Institute for Defense Analyses in carrying out the machine computations.

characteristic turbulence scale l. If the scale $l > (C_D A)^{1/2}$, then the energy builds up in a few scale lengths and then decays rather rapidly. If, however, $l < (C_D A)^{1/2}$, then the energy increase is very slight, and the decay is rather slow. For $l = 0.1(C_D A)^{1/2}$, the energy is practically constant. The asymptotic decay for $\xi \gg 1$ is, in every case, proportional to ξ^{-2}.

These results are not inconsistent with the currently accepted ideas about turbulent energy decay. The persistence of the large eddies is rather clearly brought out; whereas, at the same time, the small eddies show a tendency to hold on to their energy, as predicted by the Kolmogoroff hypothesis. The implications of the results for the calculation of the radar cross section of turbulent plasmas is the following: if we accept the hypothesis that

$$\langle n_e^2(\kappa_i, \xi) \rangle \propto \langle N_e(\xi)\rangle^2 E(\kappa_i, \xi)$$

where n_e are the fluctuations in electron density, $\langle N_e \rangle$ is the average electron density, and $E(\kappa_i, \xi)$ is the turbulent energy of eddies in the wave number interval $\Delta\kappa(\kappa_i)$, then it would appear that the Kolmogoroff assumption $E(\kappa_i, \xi) \approx$ const is quite acceptable for wavelengths that are rather small as compared with the characteristic dimensions of the body. For wavelengths which are longer than $(C_D A)^{1/2}$, it is probably necessary to take into account the fact that the turbulent energy is not constant.

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Combined Forced and Free Convection Channel Flows in Magnetohydrodynamics

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Nomenclature

B = induced magnetic field
B_0 = applied magnetic field
g = gravitational acceleration
i = $(-1)^{1/2}$
L = half-width of channel
p = pressure
T, T^* = temperature
T_{w0} = wall temperature at $z = 0$
v = velocity
α = thermal diffusivity
β = volumetric expansion coefficient
θ^* = temperature difference, $T^* - T_{w0}$
μ = magnetic permeability
ν = kinematic viscosity
ρ = density
σ = electrical conductivity

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Introduction

Considerable work has been done recently on problems dealing with heat transfer in electrically conducting fluids in the presence of electromagnetic fields. An extensive review for problems of this type is given in Ref. 1. This note deals with the combined forced and free convection of an electrically conducting fluid flowing inside an open-ended vertical channel formed by two identical parallel plates in the presence of a uniform transverse magnetic field. Solutions are obtained for the case when the surface temperatures of the plates vary linearly along the vertical distance. Such configuration in the magnetic field-free case was discussed by Ostrach.[2, 3]

When the vertical temperature gradient is taken to be negative (which is the case of heating from below), there appears a critical Rayleigh number at which the solution of the velocity becomes infinite. This was attributed[3] to the appearance of instability as it would occur for a horizontal layer heated from below. The influence of the magnetic field in the latter case has been studied extensively.[4, 5] The conclusion is that the magnetic field has a stabilizing effect. In the present configuration of vertical parallel plates, it is found that the magnetic field has the same effect.

Basic Equations and Their Solutions

For the fully developed laminar flow in a uniform transverse magnetic field, the velocity and induced magnetic field have only a vertical component, and all of the physical quantities except temperature and pressure are independent of the vertical coordinate z. Furthermore, the temperature of the fluid can be expressed as

$$T = T^*(y) + Nz \tag{1}$$

where N is the vertical temperature gradient, and y is the horizontal coordinate normal to the plates.

Under the conditions stated, the continuity equation is identically satisfied, and the momentum equations in the y and z component are

$$(\partial p/\partial y) + (1/\mu)B(dB/dy) = 0 \tag{2}$$

$$\nu(d^2v/dy^2) + (1/\rho\mu)B_0(dB/dy) + g\beta(\theta^* + Nz) - (1/\rho)(\partial p/\partial z) = 0 \tag{3}$$

and the energy and magnetic induction equations reduce to

$$Nv = \alpha(d^2\theta^*/dy^2) \tag{4}$$

$$(d^2B/dy^2) + \sigma\mu B_0(dv/dy) = 0 \tag{5}$$

In the energy equation we have neglected the viscous and Joulean dissipations.

Integrating Eq. (2) with respect to y, it gives

$$p = (B^2/2\mu) + f(z) \tag{6}$$

where $f(z)$ is an arbitrary function of z. When we substitute p into Eq. (3) and rearrange terms, it becomes

$$\nu(d^2v/dy^2) + (1/\rho\mu)B_0(dB/dy) + g\beta\theta^* = (1/\rho)(df/dz) - g\beta Nz \tag{7}$$

Since the right-hand side of Eq. (7) is a function of z only, whereas the left-hand side is a function only of y, both must be equal to a constant C_1. Thus, we write Eq. (7) as

$$\nu(d^2v/dy^2) + (1/\rho\mu)B_0(dB/dy) + g\beta\theta^* = C_1 \tag{8}$$

The constant C_1 must be related to the physics of the problem. It could be determined from either the end conditions of pressure, to which the channel is subjected, or the end conditions by the mass flow in the channel.

We introduce dimensionless variables by the substitutions $\eta = (y/L)$, $u = (Lv/\alpha)$, $t = (\theta^*/NL)$, $b = (B/B_0)$, M = Hartmann number = $B_0L(\sigma/\rho\nu)^{1/2}$, Ra = Rayleigh number =